Problem Set 5 (Solutions by Daniil Kliuev)

Problem 1 *Prove that a connected graph G is a tree if and only if any family of pairwise intersecting (that is, vertex intersecting) paths $P_1, \dots, P_k$ in G have a common vertex.*

Solution. Suppose that G is not a tree. Then G contains a cycle $v_0 \cdots v_m$, where $m \geq 3$. Then paths $P_1 = v_0v_1$, $P_2 = v_nv_0$ and $P_3 = v_1 \dots v_n$ are pairwise intersecting but do not have a common vertex.

Suppose that G is a tree, $P_1, \dots, P_k$ are pairwise intersecting paths. We will provide two proofs that $P_1, \dots, P_k$ have a common vertex.

1. Suppose that G contains n vertices. We will prove the claim by induction on n.

   Base case $n = 1$. In this case $P_1, \dots, P_k$ coincide with G, the only vertex of G is their intersection.

   Induction step $n - 1 \to n$. Since G is a tree it has a leaf v. Then $G' = G \setminus \{v\}$ is a tree on $n - 1$ vertices and $P'_1 = P_1 \setminus \{v\}$, $P'_2 = P_2 \setminus \{v\}, \dots, P'_k = P_k \setminus \{v\}$ are (possibly empty) paths.

   If $P'_1, \dots, P'_k$ are pairwise intersecting then $P'_1, \dots, P'_k$ have a common vertex $u \in V(G')$. Hence $P_1, \dots, P_k$ have a common vertex u.

   Suppose that $P'_1, \dots, P'_k$ are not pairwise intersecting. Without loss of generality we may assume that P'_1 and P'_2 do not intersect. We have $P'_1 \cap P'_2 = (P_1 \cap P_2) \setminus \{v\}$. Since $P'_1 \cap P'_2$ is empty and $P_1 \cap P_2$ is nonempty we have $P_1 \cap P_2 = \{v\}$. Let w be the only neighbor of v in G. From $P_1 \cap P_2 = \{v\}$ we deduce that P_1 or P_2 does not contain w. It follows that P_1 or P_2 consists of one vertex v. Since $P_1, \dots, P_k$ are pairwise intersecting they all contain v.

2. Cases $k = 1, 2$ are tautological, so we assume $k \geq 3$. We will prove the claim by induction on k.

   Let $k = 3$. We will prove the claim by contradiction. Suppose that $P_1 = v_1 \cdots v_k$. Since P_1, P_2, P_3 are pairwise intersecting there exists $v_i \in P_2$, $v_j \in P_3$. Without loss of generality we may assume that $i \leq j$. Let k be the maximal number such that $v_k \in P_2$, $k \leq j$. If v_k belongs to P_3 then v_k is a common vertex of P_1, P_2, P_3. Hence v_k does not belong to P_3. In particular, $v_k \neq v_j$, hence $k < j$.

   By definition of k v_{k+1} does not belong to P_2. Vertex v_k does not belong to P_3, vertex v_{k+1} does not belong to P_2, hence the edge $e = v_kv_{k+1}$ does not belong to P_2 or P_3. Therefore $P_2 \setminus \{e\} = P_2$, $P_3 \setminus \{e\} = P_3$ are still connected. On the other hand, since G is a tree, $G' = G \setminus \{e\}$ has two connected components and v_k, v_{k+1} are in different connected components of G'. The path from v_{k+1} to v_j

still exists, hence they are in the same connected component. It follows that v_k and v_j are in different connected components. Since $v_k \in P_2$, $v_j \in P_3$ paths P_2 and P_3 are in different connected components of G'. It follows that P_2 and P_3 do not intersect, a contradiction.

Let $k > 3$. Using the claim for $k-1$ we get a common vertex u of $P_2, \dots, P_k$ and a common vertex v of $P_1, P_3, P_4, \dots, P_k$. Let P' be a unique path from u to v in G. Paths $P_3, \dots, P_k$ are connected and contain u, v, hence they contain a unique path between them P'.

P_1 and P_2 intersect by assumption, P_1 and P' intersect in v, P_2 and P' intersect in u. Using the claim for $k = 3$ and paths P_1, P_2, P' we deduce that P_1, P_2, P' have common vertex w. Since $P_3, \dots, P_k$ contain P', w is a common vertex of $P_1, \dots, P_k$.

□

Problem 2 *Let A_1, A_2, A_3 be nonempty mutually-disjoint sets with $|A_i| = n_i$ for every $i \in [3]$. The **complete tripartite graph** on $A_1 \cup A_2 \cup A_3$, denoted by K_{n_1,n_2,n_3}, is the graph G with $V(G) = A_1 \cup A_2 \cup A_3$ having an edge between two vertices if and only if these vertices are not in the same set A_i. For $m, n \in \mathbb{N}$, find a formula for the number of spanning trees of the complete tripartite graph $K_{m,m,n}$.*

Solution. Assume for convenience that $A_1 = [m]$, $A_2 = [2m] \setminus [m]$, $A_3 = [2m+n] \setminus [2m]$.

Let M be the Laplacian matrix of G. We have $M_{ij} = -1$ when i, j belong to different parts of the graph. We have $M_{ii} = m + n$ when $i \in [2m]$ and $M_{ii} = 2m$ when $i \in A_3$. When $i \neq j$ belong to the same part we have $M_{ij} = 0$.

We will compute the number of spanning trees using Matrix tree theorem: it is equal to $\frac{1}{2m+n}\lambda_1 \cdots \lambda_{2m+n-1}$, where $\lambda_1, \dots, \lambda_{2m+n-1}$ are all nonzero eiganvalues of M.

Let I be the unit matrix. Consider $M - (m + n)I$. Its first m rows equal to $(0, \dots, 0, -1, \dots, -1)$, m zeroes and $m + n$ minus ones. Its rows from $m + 1$ to $2m$ equal to $(-1, \dots, -1, 0, \dots, 0, -1, \dots, -1)$, m minus ones, then m zeroes, then n minus ones. We see that the kernel of this matrix has dimension at least $2m - 2$: any vector $(v_1, \dots, v_m, v_{m+1}, \dots, v_{2m}, 0, \dots, 0)$ with $v_1 + \cdots + v_m = v_{m+1} + \cdots + v_{2m} = 0$ belongs to its kernel. We deduce that M has eigenvalue $m + n$ with multiplicity at least $2m - 2$.

Consider vector $v = (1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$, m ones, m minus ones, n zeroes. We have $(M - (m+n)I)v = mv$, hence v is an eigenvector of M with eigenvalue $2m+n$.

Consider the matrix $M - 2mI$. Its rows $2m + 1, \dots, 2m + n$ are equal to the row vector $(-1, \dots, -1, 0, \dots, 0)$, $2m$ minus ones, n zeroes. Similarly to above this means that $M - 2mI$ has kernel of dimension at least $n - 1$. Hence $2m$ is an eigenvalue of M with multiplicity at least $n - 1$.

Zero is always an eigenvalue of Laplacian, the corresponding vector is $(1, 1, \dots, 1)$.

We found $2m - 2 + 1 + n - 1 + 1 = 2m + n - 1$ eigenvalues of M. We compute the trace of M to find the last one. We have $\operatorname{tr} M = 2m(m+n) + n \cdot 2m = 2m^2 + 4mn$. The sum of eigenvalues we already found is $(2m-2)(m+n) + (2m+n) + 2m(n-1) = 2m^2 + 2mn - 2m - 2n + 2m + n + 2mn - 2m = 2m^2 + 4mn - 2m - n$. So the last eigenvalue is $2m + n$.

So the number of spanning trees is

$$\frac{1}{2m+n}(m+n)^{2m-2}(2m+n)^2(2m)^{n-1} = (2m+n)(m+n)^{2m-2}(2m)^{n-1}.$$

□

Problem 3 *For every $n \in \mathbb{N}$, let t_n be the number of trees on $[n]$. Without using Cayley's theorem (that is, $t_n = n^{n-2}$), prove that*

$$t_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k t_k (n-k) t_{n-k}.$$

Solution. We multiply by $2(n-1)$ to get $2(n-1)t_n = \sum_{k=1}^{n-1} \binom{n}{k} k t_k (n-k) t_{n-k}$.

The left-hand side counts the number of pairs $(T, \overrightarrow{e})$, where T is a tree and $\overrightarrow{e}$ is an edge $e = uv$ of T plus a choice of orientation $u \to v$. Indeed, there are t_n trees, $n-1$ edges in each of them, 2 ways to orient an edge.

We claim that the right-hand side counts the number of pairs (T_1, T_2) of rooted trees such that $V(T_1) \cap V(T_2) = \emptyset$, $V(T_1) \cup V(T_2) = [n]$. Let k be the size of T_1. Then there are $\binom{n}{k}$ ways to choose $V(T_1)$. After that $V(T_2) = [n] \setminus V(T_1)$. There are t_k ways to construct a tree on $V(T_1)$ and t_{n-k} ways to construct a tree on $V(T_2)$. It remains to choose two roots: k ways for T_1, $n-k$ ways for T_2. This proves the claim.

Now we construct a bijection between the two sets. Let $(T, \overrightarrow{e})$ be a tree with an oriented edge $e = u \to v$. Graph $T \setminus \{e\}$ is a forest with two connected components. Let T_1 be the connected component that contains u, T_2 be the connected component that contains v. These a rooted trees with roots u, v. On the other hand, if T_1, T_2 are a pair of rooted trees with roots u, v, we can obtain a tree $T = (T_1 \cup T_2) \cup \{uv\}$ with an oriented edge $u \to v$. We see that these two maps are inverse two each other.

Since the set of pairs $(T, \overrightarrow{e})$ and the set of pairs (T_1, T_2) have the same size we get $2(n-1)t_n = \sum_{k=1}^{n-1} \binom{n}{k} k t_k (n-k) t_{n-k}$. □

Problem 4 *Let T be a tree on the set of vertices $[m]$. For $n \in \mathbb{N}$ with $n > m$, in how many ways can we extend T to a tree on $[n]$?*

Solution. Let S be a tree on $[n]$ that extends T. We will construct a list of $n-m$ numbers repeating the following process $n-m$ times: take the leaf with the largest value, delete it, add the value of its parent to the list. Clearly, at each step, we are left with a tree. We claim that at each step there is a leaf with value greater than m. Indeed, if there are $m+k$ vertices and no leaves outside T then there are $m-1$ edge inside T and at least $2k$ edges outside T, giving at least $m+2k-1 > m+k-1$ edges, a contradiction. Hence this process deletes vertices from $[n]\setminus[m]$ in some order and gives back the initial tree T. Thus, at each step, we are left with a tree that is obtained as an extension of T. We also note that at the last step the parent of the corresponding leaf belongs to $[m]$, whence the last number in the obtained list belongs to $[m]$. Then we have constructed a map φ from the set $\mathscr{T}$ of trees on $[n]$ extending T to the set $\mathscr{L}$ of lists $d_1, \dots, d_{n-m}$ with $d_1, \dots, d_{n-m+1} \in [n] \setminus [m]$ and $d_{n-m} \in [m]$.

Now we show that $\varphi\colon \mathscr{T} \to \mathscr{L}$ is injective. Let D be a list in $\mathscr{L}$; that is $D = d_1, \dots, d_{n-m}$ with $d_1, \dots, d_{n-m+1} \in [n] \setminus [m]$ and $d_{n-m} \in [m]$. By construction, the degree of vertex $i > m$ is the number of times it appears in D plus one. Hence we know what are the leaves: vertices that do not appear in D. So we know what was the first deleted edge: this was an edge between d_1 and the maximum number $l_1 \in [n]\setminus[m]$ that is not in D. Repeating this gives the second deleted edge: this is an edge between d_2 and maximal number not in $\{l_1, d_2, \dots, d_{n-m}\}$. Repeating this process we get back the sequence of deleted edges. Hence there is a unique tree S extending T such that $\varphi(S) = D$.

Let us proceed to argue that φ is surjective. To do so, fix $D \in \mathscr{L}$. We define the list of leaves $l_1, \dots, l_{n-m}$ as before: l_1 is the maximum element of $[n] \setminus [m]$ that is not in D, then l_2 is the maximum element of $[n] \setminus [m]$ that is not in $\{l_1, d_2, \dots, d_{n-m}\}$, and so on until we get the last leaf l_{n-m} of our list, which is the unique element of $[n] \setminus [m]$ that is not in $\{l_1, \dots, l_{n-m-1}, d_{n-m}\}$. We will add edges to T one by one starting with $d_{n-m}l_{n-m}$. By construction $\{l_1, \dots, l_k\}$ does not intersect with $\{d_k, \dots, d_{n-m}\}$ for any $1 \le k \le n-m$. Hence when we add a new edge $l_k d_k$ to $T \cup \{l_{n-m}d_{n-m}\} \cup \dots \cup \{l_{k+1}d_{k+1}\}$ we use a new vertex l_k and do not create a cycle. It follows that $S = T \cup \{l_{n-m}d_{n-m}\} \cup \dots \cup \{l_1 d_1\}$ is a tree.

Hence we conclude that φ is a bijection and, therefore, we can extend T to a tree on $[n]$ in $|\mathscr{T}| = |\mathscr{L}| = mn^{n-m-1}$ ways. □

Problem 5 *Let G be a simple graph, and let T and T' be two spanning trees of G. Show that for each $e \in E(T)$, we can choose $e' \in E(T')$ such that $(T \setminus \{e\}) \cup \{e'\}$ and $(T' \setminus \{e'\}) \cup \{e\}$ are both spanning trees of G.*

Solution. In the case when $e \in E(T')$ we choose $e' = e$, so we assume that e does not belong to T'. In this case $T' \cup \{e\}$ contains a cycle C. Since T' does not have cycles this cycle should contain e. Let $C = u_0 \dots u_n$, $e = u_0 u_n$.

Graph $T \setminus \{e\}$ is a forest with two connected components and u_0, u_n are in different connected components. Hence we can find i such that u_i and u_{i+1} are in different connected components. Let $e' = u_i u_{i+1}$. All edges of C except e belong to T', hence e' belongs to T'.

Since e' is an edge between two different connected components of the forest $T \setminus \{e\}$ the graph $T \setminus \{e\} \cup \{e'\}$ is a tree.

Since e, e' belong to the same cycle C deleting e' from $T' \cup \{e\}$ leaves it connected: instead of e' we can use $u_i \cdots u_n u_0 \cdots u_{i-1}$ in a path that contains e'. Hence $T' \cup \{e\} \setminus \{e'\}$ is a connected graph with $|V(G)| - 1 = |V(T')| - 1$ edges, so it is a tree. □

Problem 6 *Let V_1, V_2, and V_3 be three mutually disjoint nonempty sets satisfying $|V_1| = |V_2| = |V_3| = n$, and let G be a simple graph with $V(G) = V_1 \cup V_2 \cup V_3$. Assume that every $v \in V_i$ is adjacent to exactly $n+1$ vertices in $V(G) \setminus V_i$ (v may also be adjacent to some vertices in V_i) for every $i \in [3]$. Prove that there exist $v_1, v_2, v_3 \in V(G)$ with $v_i \in V_i$ for every $i \in [3]$ such that $v_1 v_2 v_3$ is a cycle in G.*

Solution. For $1 \leq i, j \leq 3$, $i \neq j$, $v \in V_i$ we denote by $d_j(v)$ the number of neighbors of v in V_j.

Consider numbers $d_j(v)$ for all triples i, j, v, let k be their maximum. Without loss of generality we may assume that $k = d_2(v)$, where $v \in V_1$.

We have $d_2(v) + d_3(v) = n + 1$ and $d_2(v) \leq |V_2| = n$. Hence $d_3(v) \geq 1$. So there exists a vertex $w \in V_3$ connected to v. In the case when $d_2(w) + d_2(v) > n$ we can find a common neighbor u of v, w by pigeonhole principle. Hence uvw is a cycle.

So we assume that $d_2(w) + d_2(v) \leq n$. Since $d_2(w) + d_1(w) = n + 1$ we get $d_1(w) = n + 1 - d_2(w) \geq 1 + d_2(v) \geq k + 1$, a contradiction with the definition of k. □

Problem 7 *Let G be a simple graph in which every vertex has degree 3. Prove that G has a perfect matching if and only if G can be decomposed into paths of length 3 each.*

Solution. Suppose that G can be decomposed into paths of length 3 each. We take middle edge from each path.

We claim that middle edges form different paths do not have a vertex in common. Assume that this is not the case.let e_1 and e_2 be middle edges of paths P_1 and P_2 that have vertex v in common. Then the degree of v is at least 4: there are two edges incident to v in P_1 and two edges incident to v in P_2. This contradiction proves the claim.

Since different edges do not have common vertices we obtain a matching. Let n be the number of vertices in G. Then there are $\frac{3n}{2}$ edges in G. It follows that there are $\frac{n}{2}$ paths of length 3. Hence there are $\frac{n}{2}$ middle edges, so this matching in perfect.

Suppose that G has a perfect matching M. Then in graph $G \setminus M$ the degree of each vertex is two. Hence $G \setminus M$ is a union of cycles. For each vertex we will choose an edge incident to it as follows. Write each cycle C as $C = v_0 \cdots v_n$ and choose for vertex v_i edge $v_i v_{i+1}$. We note that to different vertices correspond different edges.

We construct paths of length 3 as follows. Let $e = uv$ be an element of M. To u, v correspond edges e_u, e_v in $G \setminus M$. Edges e_u, e, e_v form a path of length 3. Since to each vertex corresponds its own edge these paths do not intersect. There are $\frac{n}{2}$ of these paths, hence $E(G)$ is a union of these paths. □

Problem 8 *Let $k, n \in \mathbb{N}$ such that $k < n/2$. Let G be a bipartite graph with parts V and W satisfying the following condition:*

- *V is the set of k-subsets of $[n]$ and W is the set of $(k+1)$-subsets of $[n]$, and*
- *there is an edge between $S \in V$ and $T \in W$ if and only if $S \subseteq T$.*

Prove that V has a perfect matching into W

1. *(0.5 pts) by using Hall's theorem, and*
2. *(0.5 pts) by explicitly finding a perfect matching.*

Solution.

1. We note that the degree of each $S \in V$ equals to $n - k$ and the degree of each $T \in W$ equals to $k + 1$. Let A be a subset of V, $N(A)$ its neighborhood. There are $(n - k)|A|$ edges going from A and $(k + 1)|N(A)|$ going from $N(A)$. Since all edges from A go to $N(A)$ we have $(n - k)|A| \leq (k + 1)|N(A)|$. From $k < \frac{n}{2}$ we deduce that $(k + 1) \leq n - k$. Hence $(k + 1)|N(A)| \leq (n - k)||N(A)|$. It follows that $(n - k)|A| \leq (n - k)|N(A)|$, so $|A| \leq |N(A)|$. Since this holds for any subset A of V the conditions of Hall's theorem are satisfied and there exists a perfect matching from V to W.

2. We will modify the construction given in the StackExchange forum:

   `https://math.stackexchange.com/questions/126065/`
   `for-kn-2-construct-a-bijection-f-from-k-subsets-of-n-to-n-k-subse`

We construct a sequence of parentheses from a set S as follows: the i-th parenthesis is opening if i does not belong to S and closing if i belongs to S. Condition on k implies that there are more opening parentheses than closing. We match parentheses as follows: we go from right to left and match each closing parenthesis to the nearest non-matched opening parenthesis to the left if such exists.

In this way some parentheses will be matched and some will be unmatched. All unmatched open parentheses are to the right of unmatched close parentheses or some of them would be matched. We take leftmost unmatched open parenthesis and change it to close parenthesis. Then we convert it to the set T. We have $S \subset T$ by construction.

It remains to prove that for $S_1 \neq S_2$ we have $T_1 \neq T_2$. Suppose that there exist $S_1 \neq S_2$ such that $T_1 = T_2 = T$. By construction S_1 and S_2 differ in two elements i, j. We may assume that $i < j$. On the level of parentheses one of S_1, S_2 looks like $\cdots(\cdots)\cdots$, the other looks like $\cdots)\cdots(\cdots$. And T look like $\cdots)\cdots)\cdots$. Without loss of generality let $S_1 = \cdots)\cdots(\cdots$. Since j-th parenthesis was changed it was the leftmost unmatched opening parenthesis. It follows that all opening parentheses on places $1, \ldots, j-1$ are matched to closing parentheses.

Consider $S_2 = \cdots(\cdots)\cdots$. Start matching parentheses. Going from the rightmost parenthesis to the $j+1$-th parenthesis gives the same set of unmatched parentheses as in S_1 and two options for matched parentheses: either they are the same as in S_1 or they contain i-th parenthesis. In the second case i-th parenthesis is matched, hence S_2 does not correspond to T. So we assume that we have the same set of matched opening parentheses as in S_1.

Since all opening parentheses on places $i+1, \ldots, j-1$ were matched to closing parentheses in S_1 the number of closing parentheses in $\{i+1, \ldots, j-1\}$ is greater or equal than the number of yet-unmatched opening parentheses in $\{i+1, \ldots, j-1\}$. Hence in S_2 there are more closing parentheses in $\{i+1, \ldots, j\}$ than yet-unmatched opening parentheses. Therefore the parenthesis on place i will be matched to some closing parenthesis by construction. Hence i-th parenthesis cannot be the leftmost unmatched parenthesis, so T does not correspond to S_2, a contradiction.

□